Main topic for mid term: differentiation

<u>Def.</u> $f: S \to \mathbb{R}$ (S a subset of $\mathbb{R}$)

f differentiable at $a \in S$ (interior point of S)

if $\lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists

e.g. in practice exam:

first T/F question: f differentiable at 0

$$\overset{?}{\Rightarrow} \underbrace{\lim_{x \to 0} \frac{f(x^2) - f(0)}{|x|}}_{\shortparallel} = 0$$

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reason: $\overbrace{\lim_{x \to 0} \frac{f(x^2) - f(0)}{x^2 - 0} \cdot \underbrace{\frac{x^2}{|x|}}_{=|x|}} = f'(0) \cdot 0 = 0$ ✓

another practice exam problem:

let $f(x) = \begin{cases} x^2 & x \text{ rational} \\ x^3 & x \text{ irrational} \end{cases}$

Show: f differentiable at 0

Solution: need to show that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} \quad \text{exists!}$$

$$\frac{f(x)}{x} = \begin{cases} x & x \text{ rational} \\ x^2 & x \text{ irrational} \end{cases}$$

in either case we have $\lim_{x \to 0} \frac{f(x)}{x} = 0$

Practice exam problem 4(b): g differentiable on $\mathbb{R}$

Assume $|g(x)-g(y)| \geq |x-y|$ for every x, y in $\mathbb{R}$

prove that $|g'(x)| \geq 1$ for every x

Solution:

$$|g'(x)| = \lim_{y\to x} \left| \frac{g(y)-g(x)}{y-x} \right| = \lim_{y\to x} \frac{|g(y)-g(x)|}{|y-x|}$$

$$\geq \lim_{y\to x} \frac{|y-x|}{|y-x|} = 1$$

L'Hospital's Rule

Let $s \in \mathbb{R} \cup \{\pm\infty\}$, f, g differentiable functions (at least near s)

Assume $\lim_{x\to s} \frac{f'(x)}{g'(x)} = L$ exists

and either $\lim_{x\to s} f(x) = 0 = \lim_{x\to s} g(x)$

or $\lim_{x\to s} f(x) = \infty = \lim_{x\to s} g(x)$

$$\Rightarrow \boxed{\lim_{x\to s} \frac{f(x)}{g(x)} = L}$$

Practice exam 3(b).

$$\lim_{x \to \infty} \sqrt{x} \sin\left(\frac{1}{x}\right) = \lim_{x \to \infty} \frac{\sin(1/x)}{1/\sqrt{x}}$$

check: $\sin\left(\frac{1}{x}\right) \to 0$ for $x \to \infty$

$1/\sqrt{x} \to 0$ for $x \to \infty$

differentiable for $x \neq 0$

$\Rightarrow$ l'Hospital

$$\lim_{x \to \infty} \frac{\sin(1/x)}{\underset{\displaystyle x^{-1/2}}{\underset{\shortparallel}{1/\sqrt{x}}}} = \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right)\left(\cancel{-}\frac{1}{x^2}\right) \quad x^2}{\cancel{-}\frac{1}{2} x^{-3/2} \quad \cdot x^2}$$

$$= \lim_{x \to \infty} \frac{\cos \frac{1}{x}}{\frac{1}{2} x^{1/2}}$$

aside: $\displaystyle \lim_{x\to\infty} \left| \frac{\cos\frac{1}{x}}{\frac{1}{2}x^{1/2}} \right| \le \lim_{x\to\infty} \frac{2}{|x|^{1/2}} = 0$

$$\Rightarrow \lim_{x\to\infty} \frac{\cos\frac{1}{x}}{\frac{1}{2}x^{1/2}} = 0$$

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mean value theorem for derivatives:

$f: (a,b) \to \mathbb{R}$ differentiable, contin. on $[a,b]$

$\Rightarrow \exists\ y \in (a,b)$ s.t.

$$f'(y) = \frac{f(b) - f(a)}{b-a}$$

Taylor's Theorem

Assume f is n times differentiable in (a,b)

$c \in (a,b)$, $x \in (a,b)$

$\Rightarrow \exists\, y$ between x and c such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + R_n(x)$$

where $$R_n(x) = \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Crucial Point: To show that $f(x)$ is given by its Taylor series, we need to show that

$$\lim_{n\to\infty} R_n(x) = 0$$

Remarks ① We have seen that there are ∞-times differentiable functions f such that the Taylor series does not converge to $f(x)$ except for $x=c$

(e.g. $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$)

(here $c=0$)

② convergence can be shown if we have some estimates about the size of the derivatives.

Example: Practice problem 5(c):

given: $\boxed{|f^{(n)}(x)| \leq n!}$ for all $x \in (-1,1)$

prove: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ for all $x \in (-1,1)$.

Solution: ⚠ not enough to prove that series converges to $(-1,1)$

need to show that it converges to $f(x)$

$\Longleftrightarrow$ need to show: $\lim_{n \to \infty} R_n(x) = 0$

By Taylor's theorem: $\exists$ y between 0 and x s.t.

$$|R_n(x)| = \left| \frac{f^{(n)}(y)}{n!} (x-0)^n \right| \underset{\substack{\uparrow \\ \text{assumption}}}{\leq} \left| \frac{n!}{n!} x^n \right| \leq |x|^n \to 0$$

for $x \in (-1,1)$

$\Rightarrow$ claim.

(Remark: only works for $|x| < 1$)

Last class: Also showed how Taylor's theorem can be used to estimate values of a function:

we showed: $1 + \frac{3}{2}x < (1+x)^{3/2}$

Similarly can also be shown: $(1+x)^{3/2} < 1 + \frac{3}{2}x + \frac{3}{8}x^2$

Practice Problem 1 T/F question 4

Answer: F e.g. $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

we showed: (last class)

- all derivatives exist
- $f^{(n)}(0)=0 \quad \forall n$

$\Rightarrow$ Taylor series $= \sum_{k=0}^{\infty} 0 \cdot x^{k} = 0$ for all x

while $f(x) = e^{-1/x^2} \neq 0$ for $x \neq 0$

here: Taylor series identical zero

$f(x) \neq 0$ for $x \neq 0$

f identical zero means $f(x)=0$ for all x.